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OPTIMISATION OF HOMOGENEOUS THERMAL INSULATION LAYERS

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Abstract—The rate of heat loss from a body surrounded by a layer of homogeneous insulation is minimised by use of the Calculus of Variations.

This optimisation is with respect to variation of the outer surface of the insulation layer, subject to the isoperimetric constraint that its volume remains fixed.

Both Dirichlet and mixed boundary conditions for the temperature field on the outer surface are considered. Regular perturbation solutions in the case of small layer thickness are presented.

1. INTRODUCTION

This paper is concerned with the rate of heat loss Q from the surface S_1 of a body B through a layer of homogeneous insulation occupying a hollow domain D surrounding B. Thus the inner surface of D is S_1, while S_2 is defined as the outer surface of D. The optimisation problem investigated is that of minimising Q by variation of S_2 subject to the constraint that the total volume of D remains fixed. Dirichlet boundary conditions for the temperature field are imposed on S_1, while on S_2 both Dirichlet and mixed boundary conditions are considered.

The same type of isoperimetric problem has been studied previously. Banichuk[1] and Curtis and Walpole[2] considered the optimisation of elastic bars and shafts in torsion, and Mironov[3] studied the minimisation of the drag on a body in viscous fluid flow by variation of its shape. In this last problem the non self-adjoint nature of the Navier–Stokes equations necessitated the introduction of an adjoint velocity field to enable the derivation of an optimality condition.

A similar procedure is adopted here to allow the treatment of the case where the prescribed temperature on S_1 is not constant. The techniques of the Calculus of Variations are used to derive a necessary boundary condition holding on the optimal outer surface for both types of boundary-value problem described above. This condition is to be solved with the original boundary-value problem, the adjoint boundary-value problem, and the isoperimetric volume constraint.

After brief consideration of some simple analytic solutions for spherical B, regular perturbation methods are applied to the essentially two-dimensional case of an infinitely long prismatic body B surrounded by a thin insulation layer.

2. BOUNDARY-VALUE PROBLEMS

Let x_i denote Cartesian coordinates in D. Then the temperature field $\theta(x_i)$ within D satisfies the classical heat conduction equation for an homogeneous material thus

$$\nabla^2\theta = 0 \text{ in } D. \tag{2.1}$$

The following boundary conditions for θ are considered:

$$\theta = \Theta(q_1, r_1) \text{ (with } \Theta > 0) \text{ on } S_1; \tag{2.2}$$

with

(i) $$\theta = 0 \text{ on } S_2 \tag{2.3}$$

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or

$$\text{(ii)} \qquad \mathbf{n}\cdot\nabla\theta + \mu(\theta - \theta_A) = 0 \text{ on } S_2. \qquad (2.4)$$

Here q_1, r_1 denote curvilinear coordinates specifying a point on S_1, Θ is an infinitely differentiable function of q_1 and r_1, θ_A is the ambient temperature in the region outside the insulation layer (for boundary condition (2.4)) and μ is a constant given by

$$\mu = H^*/\kappa\rho^* c,$$

where H^* is the surface conductance, and κ, ρ^* and c are the conductivity, density, and specific heat respectively of the insulation material. The vector $\mathbf{n}$ is the unit normal out of D.

The Dirichlet condition (2.3) is somewhat simpler to treat mathematically than the mixed condition (2.4), but the latter condition is the more realistic of the two for most purposes, since it represents Newton's Law of Cooling which may be used to approximate many cooling laws (see, Carslaw and Jaeger[4]). In practice it is difficult to impose a temperature on a surface.

The boundary condition

$$\mathbf{n}\cdot\nabla\theta = \text{constant on } S_2$$

may also be imposed. It represents a constant heat flux through S_2 everywhere on S_2. The problem of minimising the heat loss from B then reduces to the well known isoperimetric problem of minimising the surface area of S_2 while keeping the volume of D fixed, and so it is not considered further here.

3. OPTIMALITY CONDITIONS

A necessary condition for minimum rate of heat loss from B through S_1 is now sought by the methods of the Calculus of Variations. The variations of the surface S_2 and temperature field θ are subject to the isoperimetric constraint

$$\int_D \mathrm{d}V = V_o = \text{const.}, \qquad (3.1)$$

and the boundary-value problem under consideration, i.e. either (i) the Dirichlet problem (2.1), (2.2) and (2.3) or (ii) the Mixed problem (2.1), (2.2) and (2.4). Denoting the optimal solution by the subscript "o", a weak variation about it is considered as follows:

$$\theta = \theta_o + \epsilon\theta_1 + o(\epsilon) \text{ in } D \cup D_o, \qquad (3.2)$$

$$\overset{(2)}{x}_i = \overset{(2)}{x}_{oi}(q_2, r_2) + \epsilon f(q_2, r_2) n_{oi}(q_2, r_2) + o(\epsilon) \text{ on } S_{20}. \qquad (3.3)$$

Here $0 < \epsilon \ll 1$, and it is assumed that θ and θ_o may be analytically continued into $D \cup D_o$ while still satisfying the appropriate boundary-value problem over D or D_o. Equation (3.3) describes a variation of the surface S_2 about the optimal surface S_{20}. The variables q_2 and r_2 are curvilinear coordinates on S_{20} such that $x_i = \overset{(2)}{x}_{oi}(q_2, r_2)$ are Cartesian coordinates of a point P on S_{20}. The quantities $\overset{(2)}{x}_i$ are the Cartesian coordinates of the point on S_2 corresponding to that point P on S_{20}. The function f is of the order of a typical length scale of B at most and is a differentiable function of q_2 and r_2. The components n_{oi} of the unit normal $\mathbf{n}_o$ on S_{20} out of D_o are also assumed to be differentiable functions of q_2, r_2. In terms of f the isoperimetric constraint (3.1) is

$$\int_{S_{20}} f\,\mathrm{d}S = 0. \qquad (3.4)$$

For any solution θ, D neighbouring the optimal solution the rate of heat loss Q from B through S_1 is given by the functional $I[\theta; D]$ where

$$Q = I[\theta; D] = \kappa \int_{S_1} \mathbf{n}\cdot\nabla\theta \, \mathrm{d}S. \tag{3.5}$$

Substituting the variations (3.2) and (3.3) into equation (3.5) and defining ΔI as $I[\theta; D] - I[\theta_o; D_o]$, the result

$$\Delta I = \epsilon\kappa \int_{S_1} \mathbf{n}\cdot\nabla\theta_1 \, \mathrm{d}S + o(\epsilon) \tag{3.6}$$

follows immediately. If S_{20} exists and satisfies the above smoothness conditions, the coefficient of ϵ in eqn (3.6) must vanish for all variations (3.2) and (3.3) satisfying condition (3.4). It remains to manipulate the r.h.s. of eqn (3.6) to obtain this necessary condition in terms of θ_o and S_{20}.

At this stage it becomes necessary to treat the Dirichlet and Mixed boundary-value problems separately.

(i) *Dirichlet problem*

If ϕ is defined as any twice differentiable function in D_o and the integral functional $L[\phi, \theta_1]$ is formed by the definition

$$L[\phi, \theta_1] \equiv \int_{D_o} \phi\nabla^2\theta_1 \, \mathrm{d}V, \tag{3.7}$$

then $L[\phi, \theta_1] = 0$, since $\nabla^2\theta_1$ vanishes in D_o because θ satisfies eqn (2.1) in D_o by assumption. By use of the Divergence Theorem this equation may be rearranged as

$$0 = \int_{\partial D_o} \left(\phi\frac{\partial\theta_1}{\partial n} - \theta_1\frac{\partial\phi}{\partial n}\right) \mathrm{d}S + \int_{D_o} \theta_1\nabla^2\phi \, \mathrm{d}V, \tag{3.8}$$

where ∂D_o is the whole boundary surface of D_o. The function ϕ is now chosen to be the solution of the "adjoint" boundary-value problem:

$$\nabla^2\phi = 0 \quad \text{in } D_o, \tag{3.9}$$

$$\phi = 0 \quad \text{on } S_{20}, \tag{3.10}$$

$$\phi = 1 \quad \text{on } S_1. \tag{3.11}$$

Equation (3.8) is then reduced to

$$0 = \int_{S_1} \frac{\partial\theta_1}{\partial n} \mathrm{d}S - \int_{S_{20}} \theta_1\frac{\partial\phi}{\partial n_o} \mathrm{d}S. \tag{3.12}$$

Substitution of the weak variations (3.2) and (3.3) into the boundary condition that θ vanishes on S_2 yields

$$\theta_1 = -f\frac{\partial\theta_o}{\partial n_o} \quad \text{on } S_{20}. \tag{3.13}$$

Use of results (3.12) and (3.13) to simplify equation (3.6) yields

$$\Delta I = -\epsilon\kappa \int_{S_{20}} f\frac{\partial\theta_o}{\partial n_o}\frac{\partial\phi}{\partial n_o} \mathrm{d}S + o(\epsilon).$$

For the optimal solution θ_o, S_{20} the coefficient of ϵ in this expression is to vanish for all variations f satisfying condition (3.4), so that θ_o, ϕ and S_{20} must satisfy the necessary condition

$$\frac{\partial \theta_o}{\partial n_o}\frac{\partial \phi}{\partial n_o} = \lambda_o = \text{const on } S_{20}, \tag{3.14}$$

as well as the boundary-value problems (2.1)–(2.3) and (3.9)–(3.11) and the isoperimetric constraint (3.1). Note that in the case where the prescribed temperature on S_1 is constant, θ_o and ϕ are linear multiples of each other, so that condition (3.14) reduces to

$$\frac{\partial \theta_o}{\partial n_o} = \text{const on } S_{20},$$

that is the flux on S_{20} is constant.

(ii) *Mixed problem*

For this problem ϕ is first defined as any twice differentiable function in D and an integral functional $M[\phi, \theta]$, given by

$$M[\phi, \theta] = \int_{S_2} \phi\{\mathbf{n}\cdot\nabla\theta + \mu(\theta - \theta_A)\}\,\mathrm{d}S, \tag{3.15}$$

is formed on S_2. Substitution of the boundary condition (2.4) then yields $M[\phi, \theta] = 0$. By means of the Divergence Theorem eqn (3.15) may be rewritten as

$$0 = \int_D (\phi\nabla^2\theta + \nabla\phi.\nabla\theta)\,\mathrm{d}V - \int_{S_1} \phi\mathbf{n}\cdot\nabla\theta\,\mathrm{d}S + \int_{S_2} \mu\phi(\theta - \theta_A)\,\mathrm{d}S. \tag{3.16}$$

This manipulation avoids the consideration of the variation of the normal derivative of θ on S_2 with varying S_2. If ϕ is now chosen to satisfy

$$\nabla^2\phi = 0 \text{ in } D \tag{3.17}$$

and

$$\phi = 1 \text{ on } S_1, \tag{3.18}$$

eqn (3.16) may be written as

$$\int_{S_1} \mathbf{n}\cdot\nabla\theta\,\mathrm{d}S = \int_D \nabla\phi.\nabla\theta\,\mathrm{d}V + \int_{S_2} \mu\phi(\theta - \theta_A)\,\mathrm{d}S.$$

Substitution of the variations (3.2) and (3.3) then yields

$$\begin{aligned}\epsilon\int_{S_1} \mathbf{n}\cdot\nabla\theta_1\,\mathrm{d}S = &-\int_{S_1} \mathbf{n}\cdot\nabla\theta_o\,\mathrm{d}S + \int_{D_o} \nabla\phi.\nabla\phi_o\,\mathrm{d}V \\ &+ \epsilon\int_{D_o} \nabla\phi.\nabla\theta_1\,\mathrm{d}V + \epsilon\int_{S_{20}} f\nabla\phi\cdot\nabla\theta_o\,\mathrm{d}S + \int_{S_{20}} \mu\phi(\theta_o - \theta_A)\,\mathrm{d}S \\ &+ \epsilon\int_{S_{20}} \mu f\left[\frac{\partial\phi}{\partial n_o}(\theta_o - \theta_A) + \phi\frac{\partial\theta_o}{\partial n_o}\right]\mathrm{d}S + \epsilon\int_{S_{20}} \mu\,\phi\,\theta_1\,\mathrm{d}S + o(\epsilon).\end{aligned}$$

The relations (2.1), (2.2), (2.4), (3.17) and (3.18) may be used to simplify this expression to

$$\begin{aligned}\int_{S_1} \mathbf{n}\cdot\nabla\theta_1\,\mathrm{d}S = &\int_{S_{20}} f\nabla\phi\cdot\nabla\theta_o\,\mathrm{d}S + \int_{S_{20}} \theta_1(\mathbf{n}_o\cdot\nabla\phi + \mu\phi)\,\mathrm{d}S \\ &+ \int_{S_{20}} f\mu\left[\frac{\partial\phi}{\partial n_o}(\theta - \theta_A) + \phi\frac{\partial\theta_o}{\partial n_o}\right]\mathrm{d}S + o(\epsilon).\end{aligned} \tag{3.19}$$

If ϕ is chosen to satisfy

$$\mathbf{n}_o \cdot \nabla\phi + \mu\phi = 0 \text{ on } S_{20}, \tag{3.20}$$

then eqn (3.19) reduces finally to

$$\int_{S_1} \mathbf{n}\cdot\nabla\theta_1 \, dS = \int_{S_{20}} f[\nabla\phi\cdot\nabla\theta_o - 2\mu^2\phi(\theta_o - \theta_A)] \, dS + o(\epsilon).$$

Substituting this result into eqn (3.6) yields

$$\Delta I = \epsilon\kappa \int_{S_{20}} f[\nabla\phi\cdot\nabla\phi_o - 2\mu^2\phi(\theta_o - \theta_A)] \, dS + o(\epsilon).$$

If θ_o, S_{20} is the optimal solution then the coefficient of ϵ in this expression is to vanish for all variations f satisfying condition (3.4), so that θ_o, ϕ and f must satisfy the necessary condition

$$\nabla\phi\cdot\nabla\phi_o - 2\mu^2\phi(\theta_o - \theta_A) = -\lambda_o = \text{const. on } S_{20}, \tag{3.21}$$

as well as the boundary-value problems (2.1), (2.2) and (2.4) and (3.17), (3.18) and (3.20), and the isoperimetric constraint (3.1).

Again it may be observed that in the case where the prescribed temperature on S_1 is constant, $\theta_o - \theta_A$ and ϕ are linear multiples of each other, so that conditions (2.4), (3.20) and (3.21) are satisfied when $\theta_o - \theta_A$ and $\mathbf{n}_o\cdot\nabla\theta_o$ are constants on S_{20} (related by condition (2.4)); that is, an isothermal surface enclosing the required volume on which the heat flux is constant (if such a surface exists) satisfies the necessary condition for optimality.

While sufficient conditions for optimality for both the above problems have not as yet been established, a solution satisfying the appropriate necessary condition as above may be compared with other solutions, so that some confidence in it as the optimal solution can be gained. This procedure has been followed for the solutions of Section 3.5. For any general shape of B the solution of either of the above optimality problems might be attempted by means of adapting existing finite element or difference methods for numerical solution of free boundary problems. The unknown constants in conditions (3.14) and (3.21) and the necessity (for general prescribed θ on S_1) of solving two boundary-value problems with one common boundary condition in conjunction appear to be new features. In the present paper attention is confined to several cases of practical interest which are amenable to solution by analytical means.

4. TWO SPECIAL SOLUTIONS

The special case where B is a sphere of radius a and the prescribed temperature on S_1 takes the constant value α is now considered for both types of boundary condition imposed on S_2. If r is defined as the radial distance of a point from the centre of B, then the surface S_1 is given by $r = a$ in each case.

(i) *Dirichlet condition*

A solution of problem (2.1)–(2.3), (3.1), (3.9)–(3.11), and (3.14) is:

$$\theta_o = \alpha a(b/r - 1)/(b-a), \quad \phi = a(b/r - 1)/(b-a),$$

$$\lambda_o = \alpha a^2/b^2(b-a)^2, \quad V_o = \frac{4}{3}\pi(b^3 - a^3).$$

The surface S_{20} is given by the sphere $r = b$ (where $b > a$) and the corresponding heat flux out of S_1 is $4\pi\alpha ab/(b-a)$.

(ii) *Mixed condition*

A similar spherically symmetric solution of problem (2.1), (2.2), (2.4), (3.1), (3.17), (3.18), (3.20) and (3.21) is as follows:

$$\theta_o = [\alpha b^{-2} + \mu\alpha(r^{-1} - b^{-1}) + \mu\theta_A(a^{-1} - r^{-1})]/[b^{-2} + \mu(a^{-1} - b^{-1})],$$
$$\phi = [b^{-2} + \mu(r^{-1} - b^{-1})]/[b^{-2} + \mu(a^{-1} - b^{-1})],$$
$$\lambda_o = \mu^2(\alpha - \theta_A)/b^4[b^{-2} + \mu(a^{-1} - b^{-1})]^2,$$
$$V_o = \frac{4}{3}\pi(b^3 - a^3).$$

The surface S_{20} is again given by the sphere $r = b$ (where $b > a$), and the corresponding heat flux out of S_1 is

$$4\pi\mu(\alpha - \theta_A)/[b^{-2} + \mu(a^{-1} - b^{-1})].$$

The above solutions correspond to those one would intuitively expect from symmetry considerations.

5. PERTURBATION SOLUTIONS

Regular perturbation methods are now applied to the essentially two-dimensional problem of an infinitely long prismatic body B. We consider cases where the area of any right cross-section A_o of D_o is small in comparison to the square of the length L of the perimeter C_1 of the corresponding right cross-section of B.

It is useful to make a change of coordinates (in the same way as Banichuk[1]) as follows. If P is a point in A_o then it has coordinates (t, s) found by dropping a perpendicular of length t from P to a point Q on C_1 at arc-length s measured in the positive sense from some reference point on C_1. The radius of curvature of C_1 at Q is denoted by $\rho(s)$, and S_{20} is given in terms of t and s by $t = h(s)$. The area of A_o is $S^{(o)}$.

(i) *Dirichlet problem*

Problem (2.1)–(2.3), (3.1), (3.9)–(3.11) and (3.14) may be first written in terms of t and s and then non-dimensionalised by writing

$$\epsilon_1 = S^{(o)}L^{-2}, \quad H = \epsilon_1 L, \quad h = H\bar{h}, \quad t = H\bar{t}, \quad s = L\bar{s}, \quad \theta_o = \Psi\bar{\theta},$$
$$\Theta = \Psi\bar{\Theta}, \quad \phi = \bar{\phi}, \quad \kappa = \kappa\cdot 1, \quad \rho = L\bar{\rho}, \quad \lambda_o = \Psi H^{-2}\bar{\lambda}.$$

Dropping overbars it becomes:

$$(T\theta_t)_t + \epsilon_1^2(T^{-1}\theta_s)_s = 0, \quad (T\phi_t)_t + \epsilon_1^2(T^{-1}\phi_s)_s = 0,$$

where $T = 1 + \epsilon_1 t\rho^{-1}$;

$$\theta = \Theta(s), \quad \phi = 1 \text{ on } t = 0,$$
$$\theta = 0, \quad \phi = 0 \text{ on } t = h(s),$$
$$\{1 + \epsilon_1^2 h_s^2(1 + \epsilon_1 h\rho^{-1})^{-1}\}\theta_t\phi_t = \lambda \text{ on } t = h(s).$$
$$\int_0^1\int_0^h T\,dt\,ds = 1. \tag{5.1}$$

The subscripts t and s denote differentiation with respect to those variables. The term in brackets on the l.h.s. of the optimality condition arises since the normal on the optimal surface $t = h(s)$ differs in general from the normal on C_1 for the same value of s.

A regular perturbation solution is now sought by writing

$$\theta = \theta^{(o)} + \epsilon_1\theta^{(1)} + \epsilon_1{}^2\theta^{(2)} + \dots,$$
$$\phi = \phi^{(o)} + \epsilon_1\phi^{(1)} + \epsilon_1{}^2\phi^{(2)} + \dots,$$
$$h = h^{(o)} + \epsilon_1 h^{(1)} + \epsilon_1{}^2 h^{(2)} + \dots,$$
$$\lambda = \lambda^{(o)} + \epsilon_1\lambda^{(1)} + \epsilon_1{}^2\lambda^{(2)} + \dots,$$

and solving the boundary-value problem (5.1) to each order in ϵ_1. To the zero-th order this problem becomes

$$\theta_{tt}^{(o)} = \phi_{tt}^{(o)} = 0 \text{ in } A_o,$$
$$\theta^{(o)} = \Theta(s), \quad \phi^{(o)} = 1 \text{ on } t = 0,$$
$$\theta^{(o)} = \phi^{(o)} = 0 \text{ on } t = h^{(o)}(s),$$
$$\theta_t^{(o)}\phi_t^{(o)} = \lambda^{(o)} \text{ on } t = h^{(o)}(s),$$
$$\int_0^1 h^{(o)}(s) = 1.$$

The zero-th order solution is

$$\theta^{(o)} = -J\{\Theta(s)\}^{1/2}t + \Theta(s),$$
$$\phi^{(o)} = -J\{\Theta(s)\}^{-1/2}t + 1,$$
$$h^{(o)} = \{\Theta(s)\}^{1/2}J^{-1}, \quad \lambda^{(o)} = J^2,$$

where

$$J = \int_0^1 \{\Theta(s)\}^{1/2}\, ds.$$

Hence to the lowest order the optimal thickness is proportional to the square root of the temperature difference between the inner and outer surfaces, and does not depend on the radius of curvature of C_1. The above answer can also be obtained by making the appproximate assumption that the heat flux at any point on S_1 is proportional to $\Theta(s)(h(s))^{-1}$.

The first-order problem is

$$\theta_{tt}^{(1)} = -\theta_t^{(o)}\rho^{-1}, \quad \phi_{tt}^{(1)} = -\phi_t^{(o)}\rho^{-1},$$
$$\theta^{(1)} = \phi^{(1)} = 0 \text{ on } t = 0,$$
$$\theta^{(1)} = -h^{(1)}\theta_t^{(o)}, \quad \phi^{(1)} = -h^{(1)}\phi_t^{(o)} \text{ on } t = h^{(o)},$$
$$\theta_t^{(o)}\phi_t^{(1)} + \theta_t^{(1)}\phi_t^{(o)} = \lambda^{(1)} \text{ on } t = h^{(o)},$$
$$\int_0^1 \left(h^{(1)} + \frac{1}{2}(h^{(o)})^2\rho^{-1}\right) ds = 0.$$

This has the solution

$$\theta^{(1)} = \frac{1}{2}J\,\Theta^{1/2}t^2\rho^{-1} - \Theta\, t\rho^{-1},$$

$$\phi^{(1)} = \frac{1}{2}J\,\Theta^{-1/2}t^2\rho^{-1} - t\rho^{-1},$$

$$h^{(1)} = -\frac{1}{2}\Theta\, J^{-2}\rho^{-1}, \quad \lambda^{(1)} = 0.$$

Thus to the first order the optimal non-dimensional solution for h is

$$h(s) = \{\Theta(s)\}^{1/2}J^{-1} - \frac{1}{2}\epsilon_1\Theta(s)J^{-2}\rho^{-1}(s) + o(\epsilon_1). \tag{5.2}$$

The non-dimensional minimum rate of heat loss from B per unit length in a direction parallel to the generators of the prism B is

$$\epsilon_1^{-1}J^2 + \int_0^1 \Theta(s)\rho^{-1}(s)\,ds + O(\epsilon_1). \tag{5.3}$$

It may be observed from eqn (5.2) that the optimal thickness increases with increasing ρ. Result (5.3) may be compared with the value of the flux corresponding to the solution where $h(s)$ is everywhere constant (such that the right cross-sectional area of D is still $S^{(o)}$). To the zero-th order this (non-dimensional) value is

$$\epsilon_1^{-1}\int_0^1 \Theta(s)\,ds,$$

which is greater or equal to $\epsilon_1^{-1}J^2$ with equality if and only if $\Theta(s)$ is a constant by Schwarz's Lemma. In this special case it may easily be shown that the heat flux values corresponding to the two above solutions are the same up to $O(\epsilon_1^{0})$. The perturbation procedure may be continued to higher orders. In particular the solutions up to second order for the optimal and "constant $h(s)$" solutions can be compared for the case of constant $\Theta(s)$ on S_1, confirming that the greater heat loss corresponds to the latter solution

In conclusion the optimal solutions for $h(s)$ and the rate of heat loss per unit length along B are presented in terms of the original dimensional variables:

$$h(s) = S^{(o)}\{\Theta(s)\}^{1/2}J^{-1} - \frac{1}{2}\{S^{(o)}\}^2\Theta(s)J^{-2}\rho^{-1}(s) + O(\{S^{(o)}\}^3L^{-5}),$$

$$Q = \kappa J^2\{S^{(o)}\}^{-1} + \kappa\int_0^L \Theta(s)\rho^{-1}(s)\,ds + O(S^{(o)}\kappa\Psi L^{-2}),$$

where

$$J = \int_0^L \{\Theta(s)\}^{1/2}\,ds.$$

(ii) *Mixed problem*

As for the Dirichlet Problem, problem (2.1), (2.2), (2.4), (3.1)–(3.18), (3.20) and (3.21) may be rewritten in terms of t and s and then non-dimensionalised by writing

$$\epsilon_1 = S^{(o)}L^{-2},\quad H = \epsilon_1 L,\quad h = H\bar{h},\quad t = H\bar{t},\quad s = L\bar{s},$$
$$\theta_o = \Psi\bar{\theta},\quad \theta_A = \bar{\Psi}\bar{\theta}_A,\quad \Theta = \Psi\bar{\Theta},\quad \phi = \bar{\phi},\quad \kappa = \kappa\cdot 1,$$
$$\rho = L\bar{\rho},\quad \lambda_o = \Psi H^{-2}\bar{\lambda},\quad \mu = H^{-1}\bar{\mu}.$$

Dropping overbars it becomes:

$$(T\theta_t)_t + \epsilon_1^2(T^{-1}\theta_s)_s = 0,\quad (T\phi_t) + \epsilon_1^2(T^{-1}\phi_s)_s = 0,$$
$$T = 1 + \epsilon_1 t/\rho,$$
$$\theta = \Theta(s),\quad \phi = 1 \text{ on } t = 0,$$
$$(1 + \epsilon_1^2 h_s^2)^{-1/2}(\theta_t + \epsilon_1^2 h_s\theta_s) + \mu(\theta - \theta_A) = 0 \text{ on } t = h,$$
$$(1 + \epsilon_1^2 h_s^2)^{-1/2}(\phi_t + \epsilon_1^2 h_s\phi_s) + \mu\phi = 0 \text{ on } t = h,$$
$$\theta_t\phi_t + \epsilon_1^2(1 + \epsilon_1 h/\rho)^{-2}\theta_s\phi_s - 2\mu^2(\theta - \theta_A)\phi = -\lambda \text{ on } t = h,$$
$$\int_0^1\int_0^h T\,dt\,ds = 1. \tag{5.4}$$

The terms in h_s again arise because the normals to the optimal surface $t = h(s)$ and C_1 are not in general the same for a given value of s.

A perturbation solution is sought by writing

$$\theta = \theta^{(o)} + \epsilon_1 \theta^{(1)} + \ldots,$$
$$\phi = \phi^{(o)} + \epsilon_1 \phi^{(1)} + \ldots,$$
$$h = h^{(o)} + \epsilon_1 h^{(1)} + \ldots,$$
$$\lambda = \lambda^{(o)} + \epsilon_1 \lambda^{(1)} + \ldots,$$

and solving the boundary-value problem (5.4) to each order in ϵ_1. The zero-th order problem is

$$\theta^{(o)}_{tt} = \phi^{(o)}_{tt} = 0,$$
$$\theta^{(o)} = \Theta(s), \quad \phi^{(o)} = 1 \text{ on } t = 0,$$
$$\theta^{(o)}_t + \mu(\theta^{(o)} - \theta_A) = 0 \text{ on } t = h^{(o)}(s),$$
$$\phi^{(o)}_t + \mu\phi^{(o)} = 0 \text{ on } t = h^{(o)}(s),$$
$$\theta^{(o)}_t \phi^{(o)}_t - 2\mu^2(\theta^{(o)} - \theta_A)\phi^{(o)} = -\lambda^{(o)} \text{ on } t = h^{(o)}(s),$$
$$\int_0^1 h^{(o)}(s)\,\mathrm{d}s = 1.$$

This has the solution

$$\theta^{(o)} = -\{\Theta(s) - \theta_A\}^{1/2} Jt(1+\mu^{-1})^{-1} + \Theta(s),$$
$$\phi^{(o)} = -\{\Theta(s) - \theta_A\}^{-1/2} Jt(1+\mu^{-1})^{-1} + 1,$$
$$h^{(o)} = -\mu^{-1} + (1+\mu^{-1})\{\Theta(s) - \theta_A\}^{1/2} J^{-1},$$
$$\lambda^{(o)} = J^2(1+\mu^{-1})^{-2},$$

where

$$J = \int_0^1 \{\Theta(s) - \theta_A\}^{1/2}\,\mathrm{d}s.$$

Thus to zero-th order $h^{(o)}(s) + \mu^{-1}$ is linearly dependent on the square root of the difference in temperature between the inner surface S_1 and the region exterior to the body and insulation layer.

The first order problem is

$$\theta^{(1)}_{tt} = -\theta^{(o)}_t \rho^{-1}, \quad \phi^{(1)}_{tt} = -\phi^{(o)}_t \rho^{-1},$$
$$\theta^{(1)} = 0, \quad \phi^{(1)} = 0 \text{ on } t = 0,$$
$$\theta^{(1)}_t = -\mu(\theta^{(1)} + h^{(1)}\theta^{(o)}_t) \text{ on } t = h^{(o)},$$
$$\phi^{(1)}_t = -\mu(\phi^{(1)} + h^{(1)}\phi^{(o)}_t) \text{ on } t = h^{(o)},$$
$$\theta^{(o)}_t \phi^{(1)}_t + \theta^{(1)}_t \phi^{(o)}_t - 2\mu^2\{\phi^{(o)}\theta^{(1)} + \phi^{(1)}(\theta^{(o)} - \theta_A)\} - 2h^{(1)}\mu^2\{\phi^{(o)}_t(\theta^{(o)} - \theta_A) + \phi^{(o)}\theta^{(o)}_t\}$$
$$= -\lambda^{(1)} \text{on } t = h^{(o)},$$
$$\int_0^1 \left\{h^{(1)} + \frac{1}{2}(h^{(o)})^2 \rho^{-1}\right\} \mathrm{d}s = 0.$$

The first order non-dimensional solution is of the form

$$\theta^{(1)} = -\frac{1}{2}A_o(s)t^2\rho^{-1} + B_1(s)t,$$

$$\phi^{(1)} = -\frac{1}{2}C_o(s)t^2\rho^{-1} + D_1(s)t,$$

$$\lambda^{(1)} = -(2R + M)\mu^3J^2N^{-1}(1+\mu)^{-2},$$

$$h^{(1)} = [\rho^{-1}(h^{(o)})^2(3+2\mu h^{(o)}) - N^{-1}(2R+M)]/2(2\mu h^{(o)} - 1).$$

Here

$$R = \int_0^1 \frac{1}{2}(h^{(o)})^2\rho^{-1}\,\mathrm{d}s,$$

$$M = \int_0^1 \rho^{-1}(h^{(o)})^2(3+2\mu h^{(o)})(2\mu h^{(o)} - 1)^{-1}\,\mathrm{d}s,$$

and

$$N = \int_0^1 (2\mu h^{(o)} - 1)^{-1}\,\mathrm{d}s.$$

The functions $A_o(s)$, $C_o(s)$, $B_1(s)$ and $D_1(s)$ are given by

$$A_o(s) = -\{\Theta(s) - \theta_A\}(\mu^{-1} + h^{(o)})^{-1},$$

$$C_o(s) = -(\mu^{-1} + h^{(o)})^{-1},$$

$$B_1(s) = A_o(s)\left[\frac{1}{2}h^{(o)}\rho^{-1}(2+\mu h^{(o)}) - \mu^{-1}h^{(1)}\right](1+\mu h^{(o)})^{-1},$$

$$D_1(s) = C_o(s)\left[\frac{1}{2}h^{(o)}\rho^{-1}(2+\mu h^{(o)}) - \mu^{-1}h^{(1)}\right](1+\mu h^{(o)})^{-1}.$$

It follows that the optimal flux per unit length along B in non-dimensional form is given by

$$\epsilon_1^{-1}J^2(1+\mu^{-1})^{-1} - \int_0^1 B_1(s)\,\mathrm{d}s + O(\epsilon_1).$$

With substantial labour the solution may be continued to higher orders, the procedure becoming increasingly lengthy. As for the Dirichlet problem, the $O(\epsilon_1^{-1})$ term does not depend on the curvature ρ of S_1, which enters at the next order. Comparisons with the constant thickness solution, analogous to those described above for the Dirichlet problem, have been made up to the first order solution with similar outcomes.

For completeness the dimensional expressions for the thickness and the rate of heat loss per unit length Q are now presented:

$$h(s) = h^{(o)} + S^{(o)}L^{-2}h^{(1)} + O((S^{(o)})^3L^{-5}),$$

where

$$h^{(o)}(s) = -\mu^{-1} + (S^{(o)} + L\mu^{-1})\{\Theta(s) - \theta_A\}^{1/2}J^{-1}$$

and

$$h^{(1)}(s) = \frac{1}{2}L^2(S^{(o)})^{-1}\{\rho^{-1}(h^{(o)})^2(3+2\mu h^{(o)}) - N^{-1}(2R+M)\}(2\mu h^{(o)} - 1)^{-1},$$

$$Q = \kappa J^2(S^{(o)} + L\mu^{-1})^{-1} + \kappa \int_0^L \mu\{\Theta(s) - \theta_A\}(1 + \mu h^{(o)})^{-1}\left[\frac{1}{2}h^{(o)}\rho^{-1}(2 + \mu h^{(o)}) - S^{(o)}\mu h^{(1)}L^{-2}\right] ds$$
$$+ O(\kappa\Psi S^{(o)}L^{-2}).$$

Here J, R, M and N are the dimensional forms of the integrals introduced above.

6. CONCLUSIONS

The problem of minimising the rate of loss of heat from a body surrounded by an insulation layer of given volume by optimally shaping that layer has been approached by means of the techniques of the Calculus of Variations extended to variable domains. The introduction of the appropriate "adjoint" boundary-value problem for each type of boundary condition considered has enabled the derivation of a necessary (transversality) condition holding on the optimal surface in each case. The optimal solution is then in principle given by solving the governing heat-flow and adjoint boundary-value problems together with this necessary condition and the isoperimetric volume constraint.

Solutions have been derived for special cases where the body is spherical and the temperature on its surface is constant, and also for the two-dimensional case with layer thickness small in comparison to a typical length scale of the body. In the general case a numerical solution procedure would appear to be required. All surfaces have been assumed sufficiently smooth, and the temperature on the body surface has been assumed differentiable on that surface. Other cases are on interest—for example a cube has a discontinuous normal, and there may be situations where the temperature on S_1 is not differentiable. The perturbation procedure of Section 5 is not valid in these cases, and it is hoped to investigate them further.

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